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Hamiltonian structures of the multi-boson KP hierarchies, abelianization and lattice formulation

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Abstract

We present a new form of the multi-boson reduction of KP hierarchy with Lax operator written in terms of boson fields abelianizing the second Hamiltonian structure. This extends the classical Miura transformation and the Kupershmidt-Wilson theorem from the (m)KdV to the KP case. A remarkable relationship is uncovered between the higher Hamiltonian structures and the corresponding Miura transformations of KP hierarchy, on one hand, and the discrete integrable models living on *refinements* of the original lattice connected with the underlying multi-matrix models, on the other hand. For the second KP Hamiltonian structure, worked out in details, this amounts to finding a series of representations of the nonlinear $\hat{\mathbf{W}}_\infty$ algebra in terms of arbitrary finite number of canonical pairs of free fields.

1. Introduction

Multi-boson Kadomtsev-Petviashvili (KP) hierarchies are integrable systems of a very unique structure. Since their appearance as generalizations of two- and four-boson KP hierarchies [1-6] in the context of the matrix models of strings [7-10] several of their intriguing properties have been revealed and studied.

Multi-boson KP hierarchies are consistent *Poisson reductions* of the standard full (infinitely-many-field) KP hierarchy within the R-matrix scheme [11]. Let us note, that within the Lax formulation of integrable Hamiltonian systems, restrictions of the pertinent Lax operator to a submanifold *do not* necessarily lead to consistent restrictions of the corresponding Poisson structures [12]. Thus, the proof of consistency of the Poisson reductions is a necessary step in the construction of multi-boson KP hierarchies. In Ref.[11] the first KP Hamiltonian structure was formulated in terms of Darboux-Poisson coordinate pairs, i.e., canonical pairs of

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free fields. This resulted in a series of representations of $\mathbf{W}_{1+\infty}$ algebra made out of arbitrary even number of free boson fields.

In this letter we first present a new formulation of the multi-boson reduction of KP hierarchy in terms of a different set of boson fields abelianizing the second Hamiltonian structure. This extends the classical Miura transformation and the Kupershmidt-Wilson theorem [13] from the mKdV (modified Korteweg-de Vries) to the KP case. Within the KP context, it is a generalization of the Miura mapping between linear and quadratic two-boson KP hierarchies [17] to arbitrary multi-boson KP hierarchies.

One of the unique features of multi-boson KP hierarchies is their connection to the Toda lattices originating from the multi-matrix models [7,8], and the corresponding invariance under discrete symmetries [14–16]. The discrete symmetries, being implemented by a similarity transformation of the Lax operator, are canonical transformations leaving all the Hamiltonian structures form-invariant [16].

In this letter we next exhibit a remarkable relationship between the higher Hamiltonian structures and the corresponding Miura transformations of KP hierarchy, on one hand, and the discrete Toda-like integrable models living on *refinements* of the original lattice connected with the underlying multi-matrix models, on the other hand. We show that the connection with the lattice integrable models can be used to completely characterize the higher KP Hamiltonian structures in terms of the Darboux-Poisson canonical pairs of free fields. As is here explained, there exists an explicit and simple link between a set of sub-lattice (refined-lattice) spectral equations and the Lax operators of the multi-boson KP hierarchies expressed in terms of Poisson-abelian fields with respect to the given Hamiltonian structure. The fact, that the KP Hamiltonian structures turn out to be correlated with the lattice spacing of the corresponding discrete integrable systems, points to the relevance of the notion of the discrete lattice formulation for the discussion of the origin of the multi-hamiltonian structures in continuum integrable models.

Among other results we are obtaining here, is a generalization of the two-boson realization of the nonlinear, (i.e., non-Lie) $\hat{\mathbf{W}}_\infty$ algebra [18] to a series of $\hat{\mathbf{W}}_\infty$ representations in terms of arbitrary even number of ordinary free bosonic fields[6]. The latter algebra plays an important rôle as a “hidden” symmetry algebra in string-theory-inspired models with black hole solutions [18].

2. Some known basic results on reduced KP hierarchies

To set the scene we start with a brief recapitulation of some basic properties of the two-boson KP hierarchy, as well as of multi-boson KP hierarchies w.r.t. the first Hamiltonian structure, emphasizing features relevant for the present work.

2.1. Two-boson KP hierarchy

We first consider truncated elements of KP hierarchy of the type $L_{ab} = D + a\,(D-b)^{-1}$, where $D = \partial/\partial x$ and where we have introduced two Bose currents (a,b) [5]. The Lax operator can be cast in the standard form $L_{ab} = D + \sum_{n=0}^{\infty} w_n D^{-1-n}$ with coefficients $w_n = (-1)^n a(D-b)^n \cdot 1$ written in terms of the Faá di Bruno polynomials. A calculation of the Poisson bracket structures using definitions[7]:

$$\{\langle L|\, X\rangle\,,\,\langle L|\, Y\rangle\}_1 = -\langle L\,|\,[X,Y]\rangle \tag{1}$$

[6] These representations of $\hat{\mathbf{W}}_\infty$ are *not* equivalent to the representations proposed in [19] which were constructed in terms of odd number of scalar fields with alternating signatures.

[7] Here and below the following notations are used. $\mathrm{Tr}_A Z \equiv \int dx\,\mathrm{Res}Z = \int dx\, Z_{-1}(x)$ is the Adler trace for arbitrary pseudo-differential operator $Z = \sum_{k\geq -\infty} Z_k(x) D^k$, and the subscripts $\pm$ in $Z_\pm$ denote taking the purely differential or the purely pseudo-differential part of Z, respectively.

$$\{ \langle L|\, X\rangle \,,\, \langle L|\, Y\rangle \}_2 = \mathrm{Tr}_A \left((LX)_+ LY - (XL)_+ YL \right) + \int dx\, \mathrm{Res}\Big([\, L \,,\, X \,] \Big) \partial^{-1} \mathrm{Res}\Big([\, L \,,\, Y \,] \Big) \tag{2}$$

yields the first bracket structure of the two-boson (a, b) system to be given by $\{ a(x) \,,\, b(y) \}_1 = -\delta'(x-y)$ and zero otherwise. This leads the coefficients w_n of L_{ab} to satisfy Poisson-bracket structure of the linear $\mathbf{W}_{1+\infty}$ algebra type. The second bracket structure (2) takes in this case the form:

$$\begin{aligned}
\{a(x) \,,\, b(y)\}_2 &= -b(x)\delta'(x-y) - \delta''(x-y) \\
\{a(x) \,,\, a(y)\}_2 &= -2a(x)\delta'(x-y) - a'(x)\delta(x-y) \\
\{b(x) \,,\, b(y)\}_2 &= -2\,\delta'(x-y)
\end{aligned} \tag{3}$$

and based on this bracket w_n satisfy the $\hat{\mathbf{W}}_\infty$ algebra.

The above two-boson hierarchy is gauge-equivalent to the model based on the pseudo-differential operator [4]:

$$L_{ce} = (D-e)\,(D-c)\,(D-e-c)^{-1} = D + \left(e' + ec\right)(D-e-c)^{-1} \tag{4}$$

The Miura-like connection between these hierarchies generalizes the usual Miura transformation between one-bose KdV and mKdV structures and takes a form [17]:

$$a = e' + ec, \quad b = e + c \tag{5}$$

This Miura transformation can easily be seen to abelianize the second bracket (3), meaning that

$$\{ e\,(x) \,,\, c\,(y) \}_2 = -\delta'(x-y) \tag{6}$$

and a, b as given by (5) satisfy (3) by construction.

The above structures naturally appear in connection with the Toda and Volterra lattice hierarchies [20]. Consider namely the spectral equation (here $\partial \equiv \partial/\partial t_{1,1}$ where $t_{1,1}$ denotes the first evolution parameter):

$$\partial \Psi_n = \Psi_{n+1} + a_0(n)\Psi_n \tag{7}$$

$$\lambda \Psi_n = \Psi_{n+1} + a_0(n)\Psi_n + a_1(n)\Psi_{n-1} \tag{8}$$

We can cast (8) in the form $\lambda\Psi_n = L_n^{(1)}\Psi_n$ with $L_n^{(1)} \equiv \partial + a_1(n)\,(\partial - a_0(n-1))^{-1}$. Connection to the continuous hierarchy is now established by setting $a_1(n) = a$ and $a_0(n-1) = b$. The Miura transformed hierarchy described by (4) can be associated with "square-root" lattice of the original Toda lattice system of (8):

$$\lambda^{1/2}\, \widetilde{\Psi}_{n+1/2} = \Psi_{n+1} + \mathcal{A}_{n+1}\Psi_n, \quad \lambda^{1/2}\, \Psi_n = \widetilde{\Psi}_{n+1/2} + \mathcal{B}_n \widetilde{\Psi}_{n-1/2} \tag{9}$$

which defines the Volterra chain equations [20]. Also (9) can be cast into $\lambda\Psi_n = L_n^{(1)}\Psi_n$ form with

$$L_n^{(1)} = (\partial - \mathcal{A}_n)\left(\partial - \mathcal{B}_{n-1}\right)\left(\partial - \mathcal{B}_{n-1} - \mathcal{A}_n\right)^{-1} \tag{10}$$

which upon identification $\mathcal{A}_n = e, \mathcal{B}_{n-1} = c$ agrees with (4). Furthermore using one of the Volterra equations $\partial\mathcal{A}_n = \mathcal{A}_n(\mathcal{B}_n - \mathcal{B}_{n-1})$ we can rewrite (10) as

$$L_n^{(1)} = (\partial - \mathcal{A}_n)\,(\partial - \mathcal{B}_n - \mathcal{A}_n)^{-1}\,(\partial - \mathcal{B}_n) = \partial + \mathcal{B}_n\,(\partial - \mathcal{B}_n - \mathcal{A}_n)^{-1}\,\mathcal{A}_n \tag{11}$$

which upon identification $\mathcal{B}_n = \bar{\jmath}$, $\mathcal{A}_n = \jmath$ takes the form $L = D + \bar{\jmath}\,(D - \jmath - \bar{\jmath})^{-1}\,\jmath$ in which the so-called quadratic two-boson KP hierarchy appeared in connection with $SL(2,\mathbb{R})/U(1)$ coset model [4].

The above two simple 2-boson models will be generalized in the next two sections to the arbitrary multi-boson KP hierarchies.

2.2. Multi-boson KP hierarchy: the first bracket

Let us now write the generalization of the two-boson Lax L_{ab} to the arbitrary $2M$-field Lax operator [9]:

$$L_M = D + \sum_{l=1}^{M} A_l^{(M)} \left(D - B_l^{(M)}\right)^{-1} \left(D - B_{l+1}^{(M)}\right)^{-1} \cdots \left(D - B_M^{(M)}\right)^{-1} \tag{12}$$

The main result of the investigation presented in [11] is contained in the following:

Proposition. The $2M$-field Lax operators (12) are consistent Poisson reductions of the full KP Lax operator for any $M = 1, 2, 3, \ldots$.

The proof was based on the recursive formula valid for arbitrary $M = 2, 3, \ldots$

$$\begin{aligned} &L_M \equiv L_M(a,b) \equiv L_M(a_1, b_1; \ldots; a_M, b_M) \\ &L_M = e^{\int b_M}\left[b_M + (a_M - a_{M-1})D^{-1} + DL_{M-1}D^{-1}\right]e^{-\int b_M} \end{aligned} \tag{13}$$

which describes the $2M$-field Lax operators in terms of the boson fields $(a_r, b_r)_{r=1}^{M}$ spanning Heisenberg Poisson bracket algebra:

$$\{a_r(x), b_s(y)\}_{P'} = -\delta_{rs}\partial_x\delta(x-y) \tag{14}$$

In other words, we proved that the first Poisson bracket structure for L_M from (12) is given by

$$\Big\{\langle L_M \mid X\rangle \, , \langle L_M \mid Y\rangle\Big\}_{P'} = -\langle L_M \mid [X, Y]\rangle \tag{15}$$

where X, Y are arbitrary fixed elements of the algebra of pseudo-differential operators and $\langle\cdot|\cdot\rangle = \mathrm{Tr}_A(\cdot\,\cdot)$ indicates the Adler bilinear pairing. The subscript P' in (15) indicates that the constituents of $L_M(a,b)$ satisfy (14).

As a result of (13) the coefficient fields of the Lax operator (12) satisfy themselves the recursion relations

$$\begin{aligned} &A_M^{(M)} = a_M, \quad B_M^{(M)} = b_M, \quad B_l^{(M)} = b_M + B_l^{(M-1)} \quad (l = 1, 2, \ldots, M-1) \\ &A_1^{(M)} = \left(\partial + B_1^{(M-1)}\right)A_1^{(M-1)}, \quad A_l^{(M)} = A_{l-1}^{(M-1)} + \left(\partial + B_l^{(M-1)}\right)A_l^{(M-1)} \quad (l = 2, \ldots, M-1) \end{aligned} \tag{16}$$

These recursion relations can be easily solved in terms of the free fields $(a_r, b_r)_{r=1}^{M}$ from (14) to yield

$$B_l^{(M)} = \sum_{s=l}^{M} b_s, \quad A_M^{(M)} = a_M \tag{17}$$

$$A_{M-r}^{(M)} = \sum_{n_r=r}^{M-1} \cdots \sum_{n_2=2}^{n_3-1} \sum_{n_1=1}^{n_2-1} \left(\partial + b_{n_r} + \cdots + b_{n_r-r+1}\right) \cdots \left(\partial + b_{n_2} + b_{n_2-1}\right)\left(\partial + b_{n_1}\right) a_{n_1} \tag{18}$$

3. Multi-boson KP hierarchy: the second bracket

3.1. Generalized Miura transformation for multi-boson KP hierarchies

The generalized Miura transformation for multi-boson KP hierarchies, which we are going to construct in this section, can be viewed as "abelianization" of the *second* KP Hamiltonian structure (2), i.e., expressing the coefficient fields of the pertinent KP Lax operator in terms of canonical pairs of "Darboux" fields. The explicit construction relies on the following recurrence relation for multi-boson KP Lax operators:

$$L_M \equiv L_M(c,e) \equiv L_M\left(c_1,e_1;\ldots;c_M,e_M\right)$$

$$L_M = e^{\int c_M}\left(D + c_M - e_M\right) L_{M-1}\left(D - e_M\right)^{-1} e^{-\int c_M} \tag{19}$$

$$M = 1,2\ldots,\quad L_0 \equiv D$$

$$\{c_k(x)\,,\,e_l(y)\} = -\delta_{kl}\partial_x\delta(x-y),\quad k,l = 1,2,\ldots,M \tag{20}$$

As will be shown below, the pairs $(c_r,e_r)_{r=1}^{M}$ are the "Darboux" canonical pairs for the second KP bracket for arbitrary M. This defines a sequence of the multi-boson KP Lax operators in terms of the Darboux-Poisson pairs with respect to the second bracket, very much like (13) defined a similar sequence of Lax operators in terms of the Darboux-Poisson pairs with respect to the first bracket.

Eq. (19) implies the following recurrence relations for the coefficient fields of (12):

$$B_k^{(M)} = B_k^{(M-1)} + c_M \quad 1 \le k \le M-1,\quad B_M^{(M)} = c_M + e_M \tag{21}$$

$$A_1^{(M)} = \left(\partial + B_1^{(M-1)} + c_M - e_M\right) A_1^{(M-1)} \tag{22}$$

$$A_k^{(M)} = A_{k-1}^{(M-1)} + \left(\partial + B_k^{(M-1)} + c_M - e_M\right) A_k^{(M-1)},\quad 2 \le k \le M-1 \tag{23}$$

$$A_M^{(M)} = A_{M-1}^{(M-1)} + \left(\partial + c_M\right) e_M \tag{24}$$

(1) Example – 2-boson KP:

$$L_1 = e^{\int c_1}\left(D + c_1 - e_1\right) D \left(D - e_1\right)^{-1} e^{-\int c_1} = D + A_1^{(1)}\left(D - B_1^{(1)}\right)^{-1} \tag{25}$$

$$A_1^{(1)} = \left(\partial + c_1\right) e_1,\quad B_1^{(1)} = c_1 + e_1 \tag{26}$$

Here we recognize the structure of the two-boson hierarchy from (4) as well the generalized Miura map (5).

(2) Example – 4-boson KP:

$$L_2 = e^{\int c_2}\left(D + c_2 - e_2\right)\left[D + A_1^{(1)}\left(D - B_1^{(1)}\right)^{-1}\right]\left(D - e_2\right)^{-1} e^{-\int c_2}$$

$$= D + A_2^{(2)}\left(D - B_2^{(2)}\right)^{-1} + A_1^{(2)}\left(D - B_1^{(2)}\right)^{-1}\left(D - B_2^{(2)}\right)^{-1} \tag{27}$$

$$A_2^{(2)} = A_1^{(1)} + \left(\partial + c_2\right) e_2 = \left(\partial + c_1\right) e_1 + \left(\partial + c_2\right) e_2 \tag{28}$$

$$A_1^{(2)} = \left(\partial + B_1^{(1)} + c_2 - e_2\right) A_1^{(1)} = \left(\partial + e_1 + c_1 + c_2 - e_2\right)\left(\partial + c_1\right) e_1 \tag{29}$$

$$B_2^{(2)} = c_2 + e_2,\quad B_1^{(2)} = B_1^{(1)} + c_2 = e_1 + c_1 + c_2 \tag{30}$$

where $A_1^{(1)}$ and $B_1^{(1)}$ are substituted with their expressions from (26). It is easy to derive a second bracket structure for the above fields directly from (20). A simple calculation gives $\{B_2^{(2)}(x)\,,\,B_2^{(2)}(y)\} = \{B_1^{(2)}(x)\,,\,B_1^{(2)}(y)\} = -2\partial_x\delta(x-y)$, $\{B_2^{(2)}(x)\,,\,B_1^{(2)}(y)\} = -\partial_x\delta(x-y)$ etc. thus reproducing the result of [10] based on Lenard relations.

From recursive relation (19) we can obtain closed expressions for the arbitrary Lax L_M for $M = 1,2,\ldots$ directly in terms of the building blocks $(c_r, e_r)_{r=1}^{M}$:

$$L_M = (D - e_M) \prod_{k=M-1}^{1} \left(D - e_k - \sum_{l=k+1}^{M} c_l \right) \left(D - \sum_{l=1}^{M} c_l \right) \prod_{k=1}^{M} \left(D - e_k - \sum_{l=k}^{M} c_l \right)^{-1} \tag{31}$$

or, equivalently, in a "dressing" form

$$L_M = \mathcal{U}_M \mathcal{U}_{M-1} \ldots \mathcal{U}_1 \, D \, \mathcal{V}_1^{-1} \ldots \mathcal{V}_{M-1}^{-1} \mathcal{V}_M^{-1} \tag{32}$$

$$\mathcal{U}_k \equiv (D - e_k)\, e^{\int c_k} \quad , \quad \mathcal{V}_k \equiv e^{\int c_k} (D - e_k) \tag{33}$$

The recurrence relations (21)–(24) can be explicitly solved in terms of the Darboux fields:

$$B_k^{(M)} = e_k + \sum_{l=k}^{M} c_l, \quad 1 \le k \le M \qquad A_M^{(M)} = \sum_{k=1}^{M} (\partial + c_k)\, e_k \tag{34}$$

$$A_k^{(M)} = \sum_{n_k=1}^{k} \left(\partial + e_{n_k} - e_{n_k+M-k} + \sum_{l_k=n_k}^{n_k+M-k} c_{l_k} \right) \sum_{n_{k-1}=1}^{n_k} \left(\partial + e_{n_{k-1}} - e_{n_{k-1}+M-1-k} + \sum_{l_{k-1}=n_{k-1}}^{n_{k-1}+M-1-k} c_{l_{k-1}} \right) \times \cdots$$
$$\times \sum_{n_2=1}^{n_3} \left(\partial + e_{n_2} - e_{n_2+1} + c_{n_2} + c_{n_2+1} \right) \sum_{n_1=1}^{n_2} \left(\partial + c_{n_1} \right) e_{n_1}, \qquad k = 1, \ldots, M-1 \tag{35}$$

The Miura-transformed form of L_M reads explicitly:

$$L_M = D + \sum_{k=1}^{\infty} U_k[(c,e)](x) D^{-k} \tag{36}$$

$$U_k[(c,e)](x) = P_{k-1}^{(1)}\,(e_M + c_M) \sum_{l=1}^{M} (\partial + c_l)\, e_l$$
$$+ \sum_{r=1}^{\min(M-1,k-1)} A_{M-r}(c,e) P_{k-1-r}^{(r+1)} \big(e_M + c_M, e_{M-1} + c_{M-1} + c_M, \ldots, e_{M-r} + \sum_{l=M-r}^{M} c_l \big) \tag{37}$$

where $A_{M-r}(c,e)$ are the same as in (35), and $P_n^{(N)}$ denote the (multiple) Faá di Bruno polynomials [11]:

$$P_n^{(N)}(B_N, B_{N-1}, \ldots, B_1) = \sum_{m_1 + \cdots + m_N = n} (-\partial + B_1)^{m_1} \cdots (-\partial + B_N)^{m_N} \cdot 1 \tag{38}$$

Now, upon substitution of (36)–(37) into (2), we obtain a series of explicit (Poisson bracket) realizations of the nonlinear $\hat{\mathbf{W}}_\infty$ algebra in terms of $2M$ bosonic fields, satisfying (20), for any $M = 1,2,\ldots$.

3.2. Consistency of multi-boson KP Poisson reduction w.r.t. the second bracket

The main result of this paper follows from the following general statement (analogue of the Kupershmidt-Wilson theorem for mKdV [13]).

Proposition. *Let $(c_r, e_r)_{r=1}^{M}$ obey the Heisenberg Poisson algebra (20). Then the reduced Lax operators L_M (32) (or (36)–(37)) satisfy the second KP Hamiltonian structure (2).*
Proof. It proceeds by induction w.r.t. M. The case $M = 1$ is the familiar example of two-boson KP hierarchy. Substituting according to (32)–(33) $L_M = \mathcal{U}_M L_{M-1} \mathcal{V}_M^{-1}$ into (2) we have

$$
\begin{aligned}
&\{ \langle L_M | X \rangle , \langle L_M | Y \rangle \} \\
&= \{ \langle L_{M-1} | \mathcal{V}_M^{-1} X \mathcal{U}_M \rangle , \langle L_{M-1} | \mathcal{V}_M^{-1} Y \mathcal{U}_M \rangle \} && (39)\\
&+ \{ \langle \mathcal{U}_M | L_{M-1} \mathcal{V}_M^{-1} X \rangle , \langle \mathcal{U}_M | L_{M-1} \mathcal{V}_M^{-1} Y \rangle \} + \{ \langle \mathcal{V}_M^{-1} | X \mathcal{U}_M L_{M-1} \rangle , \langle \mathcal{V}_M^{-1} | Y \mathcal{U}_M L_{M-1} \rangle \} && (40)\\
&+ \{ \langle \mathcal{U}_M | L_{M-1} \mathcal{V}_M^{-1} X \rangle , \langle \mathcal{V}_M^{-1} | Y \mathcal{U}_M L_{M-1} \rangle \} + \{ \langle \mathcal{V}_M^{-1} | X \mathcal{U}_M L_{M-1} \rangle , \langle \mathcal{U}_M | L_{M-1} \mathcal{V}_M^{-1} Y \rangle \} && (41)
\end{aligned}
$$

In all terms (39)–(41) in the r.h.s. of the above equation it is understood that the Poisson brackets are taken w.r.t. the left members in the angle brackets. Henceforth, for simplicity, we shall skip the subscripts M of $\mathcal{U}, \mathcal{V}, c, e$. Using the induction hypothesis and simple identities for pseudo-differential operators, the bracket (39) takes the form

$$
\begin{aligned}
(39) &= \mathrm{Tr}_A \Big(\big(L_{M-1} \mathcal{V}^{-1} X \mathcal{U}\big)_+ L_{M-1} \mathcal{V}^{-1} Y \mathcal{U} - \big(\mathcal{V}^{-1} X \mathcal{U} L_{M-1}\big)_+ \mathcal{V}^{-1} Y \mathcal{U} L_{M-1} \Big) \\
&+ \int dx\, \mathrm{Res}\Big(\big[L_{M-1} , \mathcal{V}^{-1} X \mathcal{U} \big] \Big) \partial^{-1} \mathrm{Res}\Big(\big[L_{M-1} , \mathcal{V}^{-1} Y \mathcal{U} \big] \Big) && (42)\\
&= \mathrm{Tr}_A \big((L_M X)_+ L_M Y - (X L_M)_+ Y L_M \big) && (43)\\
&- \mathrm{Tr}_A \Big[\big(\mathcal{U}^{-1} (L_M X)_+ \mathcal{U}\big)_- \big(\mathcal{U}^{-1} (L_M Y)_+ \mathcal{U}\big)_+ - \big(\mathcal{V}^{-1} (X L_M)_+ \mathcal{V}\big)_- \big(\mathcal{V}^{-1} (Y L_M)_+ \mathcal{V}\big)_+ \Big] && (44)\\
&+ \int dx\, \mathrm{Res}\Big(\mathcal{U}^{-1} L_M X \mathcal{U} - \mathcal{V}^{-1} X L_M \mathcal{V} \Big) \partial^{-1} \Big(\mathcal{U}^{-1} L_M Y \mathcal{U} - \mathcal{V}^{-1} Y L_M \mathcal{V} \Big) && (45)
\end{aligned}
$$

The following identities, valid for arbitrary pseudo-differential operator Z and arbitrary function f, will be used in the sequel

$$
\begin{aligned}
&\mathrm{Res}\left(Z - \mathcal{U}^{-1} Z \mathcal{U} \right) = \partial_x \mathrm{Res}\left((D - e)^{-1} Z \right), \quad \mathrm{Res}\left((D - f)^{-1} Z \right) = \left(e^{\int f} Z e^{-\int f} \right)_0^{(R)} \\
&\mathrm{Res}\left(Z - \mathcal{V}^{-1} Z \mathcal{V} \right) = \partial_x \mathrm{Res}\left((D - e - c)^{-1} Z \right) && (46)
\end{aligned}
$$

where the last sub/superscripts indicate taking the zero-order (c-number) part of the corresponding *right-ordered*[8] pseudo-differential operator. Using (46) the term (45) can be rewritten in the form

$$
\begin{aligned}
(45) &= \int dx\, \mathrm{Res}\Big([\, L_M , X \,] \Big) \partial^{-1} \mathrm{Res}\Big([\, L_M , Y \,] \Big) \\
&+ \int dx\, \mathrm{Res}\Big([\, L_M , X \,] \Big) \mathrm{Res}\left((D - e - c)^{-1} Y L_M - (D - E)^{-1} L_M Y \right) - \Big(X \longleftrightarrow Y \Big) \\
&+ \int dx\, \partial_x \left\{ \mathrm{Res}\left((D - e - c)^{-1} X L_M - (D - e)^{-1} L_M X \right) \right\} \\
&\times \mathrm{Res}\left((D - e - c)^{-1} Y L_M - (D - e)^{-1} L_M Y \right) && (47)
\end{aligned}
$$

[8] That means, the coefficient functions are to the right w.r.t. differential operators D.

Similarly, taking into account (46), the trace term (44) takes the form

$$
\begin{aligned}
(44) = & \int dx\, \partial_x \left\{ \mathrm{Res}\left((D-e)^{-1} L_M X \right) \right\} \mathrm{Res}\left((D-e)^{-1} L_M Y \right) \\
& - \int dx\, \partial_x \left\{ \mathrm{Res}\left((D-e-c)^{-1} X L_M \right) \right\} \mathrm{Res}\left((D-e-c)^{-1} Y L_M \right)
\end{aligned}
\tag{48}
$$

Finally, using the Heisenberg Poisson algebra (20) for (c_M, e_M), (part of assumption in the Proposition), and using again (46), the sum of the terms (40) and (41) reads

$$
\begin{aligned}
(40) + (41) = & \int dx \left[\partial_x \left\{ \mathrm{Res}\left((D-e)^{-1} L_M X \right) \right\} - \mathrm{Res}\Big([\, L_M \,,\, X \,] \Big) \right] \\
& \times \mathrm{Res}\left((D-e-c)^{-1} Y L_M - (D-e)^{-1} L_M Y \right) - \Big(X \longleftrightarrow Y \Big)
\end{aligned}
\tag{49}
$$

Thus, collecting the results (43), (47), (49) and (48), we achieve precise cancellation of the unwanted terms leaving us with the desired result:

$$
\{ \langle L_M |\, X \rangle \,,\, \langle L_M |\, Y \rangle \} = \mathrm{Tr}_A \left((L_M X)_+ L_M Y - (X L_M)_+ Y L_M \right) + \int dx \, \mathrm{Res}\Big([\, L_M \,,\, X \,] \Big) \partial^{-1} \mathrm{Res}\Big([\, L_M \,,\, Y \,] \Big)
$$

4. The sub-lattices and abelianization of higher brackets

In this section the recurrence relations presented above will be traced back to the lattice formulation based on the discrete spectral equations. We start by providing link between lattice formulation and recurrence relation in [11] amounting to abelianization of the first KP bracket.

We start with the spectral equation

$$
\lambda \Psi_n = L_n^{(N)} \Psi_n \qquad \forall n \tag{50}
$$

where

$$
L_n^{(N)} = \partial + \sum_{k=1}^{N} a_k(n) \frac{1}{\partial - a_0(n-k)} \cdots \frac{1}{\partial - a_0(n-1)} \tag{51}
$$

Multiplying $L_n^{(N)}$ on both sides by $1 = e^{\int a_0(n-1)} e^{-\int a_0(n-1)}$ we arrive at

$$
\begin{aligned}
L_n^{(N)} = & e^{\int a_0(n-1)} \Big\{ a_0(n-1) + a_1(n) \partial^{-1} \partial \\
& + \left(+ \sum_{k=2}^{N} a_k(n) \frac{1}{\partial + a_0(n-1) - a_0(n-k)} \cdots \frac{1}{\partial + a_0(n-1) - a_0(n-2)} \partial^{-1} \right) \Big\} e^{-\int a_0(n-1)}
\end{aligned}
\tag{52}
$$

Recalling the Toda equation of motion, which is a consistency condition following from (7) and (50):

$$
a_k(n) = a_k(n-1) + \Big(\partial + a_0(n-k) - a_0(n-1) \Big) a_{k-1}(n-1) \qquad k = 1, 2, \ldots, n \tag{53}
$$

and the simple identity

$$
[\, \partial a_{k-1}(n-1) - a_{k-1}(n-1) \left(a_0(n-1) - a_0(n-k) \right)] \frac{1}{\partial + a_0(n-1) - a_0(n-k)}
$$

$$= \partial \left(a_{k-1}(n-1) \frac{1}{\partial + a_0(n-1) - a_0(n-k)} \right) - a_{k-1}(n-1) \tag{54}$$

we find

$$\begin{aligned} L_n^{(N)} &= e^{\int a_0(n-1)} \Big\{ a_0(n-1) + a_1(n)\partial^{-1} \\ &+ [a_N(n-1) \frac{1}{\partial + a_0(n-1) - a_0(n-N)} \cdots \frac{1}{\partial + a_0(n-1) - a_0(n-2)} - a_1(n-1)]\partial^{-1} \\ &+ \partial[\partial + \sum_{l=1}^{N-1} a_l(n-1) \frac{1}{\partial + a_0(n-1) - a_0(n-1-l)} \cdots \frac{1}{\partial + a_0(n-1) - a_0(n-2)}]\partial^{-1} \Big\} \\ &\times e^{-\int a_0(n-1)} \end{aligned} \tag{55}$$

This connects lattice formulation to recurrence relation (13) [11]. To see it more clearly recall now expression (12) for the continuous Lax operator. Comparing with (51) we see the following correspondence:

$$A_{N-k+1}^{(N)} \sim a_k(N); \quad A_N^{(N)} = a_N \sim a_1(N); \quad B_{N-k+1}^{(N)} \sim a_0(N-k) \quad k = 1, \ldots, N \tag{56}$$

where we put $n = N$. It is now obvious that the lattice Toda equation of motion (53) corresponds to the recurrence relation (16):

$$A_l^{(N)} = A_{l-1}^{(N-1)} + \left(\partial + B_l^{(N-1)}\right) A_l^{(N-1)} \quad (l = 2, \ldots, N-1) \tag{57}$$

with $B_l^{(N)} = b_M + B_l^{(N-1)}$, $B_N^{(N)} = b_N \sim a_0(N-1)$ and the further correspondence $B_l^{(N-1)} \sim a_0(l-1) - a_0(n-1)$ established by comparing (12) with (55).

Therefore we have established relation between the lattice system with the Toda equation (53) relating functions on different sites and corresponding continuous system with recurrence relations relating different orders of reduction of KP.

We now discuss a link between "square-root" lattice formulation and recurrence relation (19). We define spectral equation:

$$\lambda^{1/2}\, \widetilde{\Psi}_{n+1/2} = \Psi_{n+1} + \mathcal{A}_{n+1}^{(0)} \Psi_n + \sum_{p=1}^{N} \mathcal{A}_{n-p+1}^{(p)} \Psi_{n-p} \tag{58}$$

$$\lambda^{1/2}\, \Psi_n = \widetilde{\Psi}_{n+1/2} + \mathcal{B}_n^{(0)} \widetilde{\Psi}_{n-1/2} \tag{59}$$

We also have time evolution equations:

$$\widetilde{\Psi}_{n+1/2} = \left(\partial - \mathcal{B}_n^{(0)} - \mathcal{A}_n^{(0)}\right) \widetilde{\Psi}_{n-1/2}; \quad \Psi_{n+1} = \left(\partial - \mathcal{B}_n^{(0)} - \mathcal{A}_{n+1}^{(0)}\right) \Psi_n \tag{60}$$

We therefore find

$$\lambda^{1/2}\, \widetilde{\Psi}_{n+1/2} = \left(\partial - \mathcal{B}_n^{(0)} + \sum_{p=1}^{N} \mathcal{A}_{n-p+1}^{(p)} (\partial - \mathcal{B}_{n-p}^{(0)} - \mathcal{A}_{n-p+1}^{(0)})^{-1} \cdots (\partial - \mathcal{B}_{n-1}^{(0)} - \mathcal{A}_n^{(0)})^{-1} \right) \Psi_n \tag{61}$$

and

$$\lambda^{1/2}\, \Psi_n = \left(\partial - \mathcal{A}_n^{(0)}\right) \widetilde{\Psi}_{n-1/2} \tag{62}$$

from the last two relations we find

$$\lambda \Psi_n = \left(\partial - \mathcal{A}_n^{(0)}\right) \left(\partial - \mathcal{B}_{n-1}^{(0)} + \sum_{p=1}^{N} \mathcal{A}_{n-p}^{(p)} (\partial - \mathcal{B}_{n-p-1}^{(0)} - \mathcal{A}_{n-p}^{(0)})^{-1} \cdots (\partial - \mathcal{B}_{n-2}^{(0)} - \mathcal{A}_{n-1}^{(0)})^{-1} \right)$$
$$\times (\partial - \mathcal{B}_{n-1}^{(0)} - \mathcal{A}_n^{(0)})^{-1} \Psi_n \tag{63}$$

This defines a Lax operator through $\lambda \Psi_n = L_n^{(N+1)} \Psi_n$ where

$$L_n^{(N+1)} = e^{\int \mathcal{B}_{n-1}^{(0)}} \left(\partial - \mathcal{A}_n^{(0)} + \mathcal{B}_{n-1}^{(0)}\right) L_n^{(N)} \left(\partial - \mathcal{A}_n^{(0)}\right)^{-1} e^{-\int \mathcal{B}_{n-1}^{(0)}} \tag{64}$$

and

$$L_n^{(N)} = \partial + \sum_{p=1}^{N} \mathcal{A}_{n-p}^{(p)} (\partial + \mathcal{B}_{n-1}^{(0)} - \mathcal{B}_{n-p-1}^{(0)} - \mathcal{A}_{n-p}^{(0)})^{-1} \cdots (\partial + \mathcal{B}_{n-1}^{(0)} - \mathcal{B}_{n-2}^{(0)} - \mathcal{A}_{n-1}^{(0)})^{-1} \tag{65}$$

establishing one to one correspondence with the recurrence relation (19).

We now introduce notion of "1/4" lattice, which is a further sub-lattice of the "square-root" lattice associated to the Volterra hierarchy. Let us first rewrite (9) in a compact notation as

$$\lambda^{1/2} \Phi_n = \Phi_{n+1} + V_n \Phi_{n-1} \tag{66}$$

where we introduced a compact notation intrinsic for the square-lattice system through $V_{2n} = \mathcal{B}_n, V_{2n-1} = \mathcal{A}_n$ and $\Phi_{2n} = \Psi_n, \Phi_{2n-1} = \widetilde{\Psi}_{n-1/2}$. It turns out that the process of "refinement" of the lattice can be continued. We associate the following spectral system:

$$\lambda^{1/4} \widetilde{\Phi}_{n+1/2} = \Phi_{n+1} - W_n \Phi_n; \quad \lambda^{1/4} \Phi_n = \widetilde{\Phi}_{n+1/2} + W_n \widetilde{\Phi}_{n-1/2} \tag{67}$$

to the "one-quarter" lattice. In (67) we have introduced the new object W_n which can be expressed through square-lattice components as $W_{2n} = \beta_n, W_{2n-1} = \alpha_n$. Combining two equations of (67) one gets again equation (66) with $V_n = -W_n W_{n-1}$. Recalling the Volterra equation $\partial V_n = V_n (V_{n+1} - V_{n-1})$ we find the evolution equation for W_n-field to be $\partial W_n = W_n^2 (W_{n+1} - W_{n-1})$ or in components

$$\partial \alpha_n = \alpha_n^2 (\beta_n - \beta_{n-1}); \quad \partial \beta_n = \beta_n^2 (\alpha_{n+1} - \alpha_n) \tag{68}$$

Relation $V_n = -W_n W_{n-1}$ translates in components to

$$\mathcal{A}_n = -\alpha_n \beta_n + \frac{\alpha_n'}{\alpha_n}; \quad \mathcal{B}_n = -\beta_n \alpha_n \tag{69}$$

which identifies the new hierarchy, connected with the "1/4"-lattice, with the so-called derivative Non-Linear Schrödinger (dNLS) hierarchy [15]. In [15] it was in fact shown that for the fields of dNLS hierarchy $\alpha_n = -q, \beta_n = -r - q'/q^2$ the third Poisson bracket structure

$$\{q(x), r(y)\}_3 = \delta'(x - y) \tag{70}$$

takes an abelian form.

5. Outlook

The theory of integrable lattice systems has a profound geometrical foundation and found recently new important physical applications. Among other results obtained in this paper, we have used the Toda lattice system and the related discrete integrable systems, living on its sub-lattices, to "coordinatize" the continuum multi-boson KP hierarchies and their Hamiltonian structures. It was shown how KP Miura transformations are related to the underlying lattice structures and how transition to the "finer" sub-lattice provides the right set of abelian field coordinates for the higher KP Hamiltonian structures. In view of the significance of higher Hamiltonian structures for the notion of integrability this emphasizes the need of a general approach to results we have obtained in this paper. A completely systematic theory should not only fully explain the link between the various lattice integrable systems and the higher Hamiltonian structures, but should also address the relation between the Toda lattice Poisson brackets (with discrete indices) and field-theoretical Poisson brackets in continuum w.r.t. the first lattice evolution parameter. We plan to address these interesting issues in the future.

In addition we hope that the established connection between the higher Hamiltonian structures and the pertinent Miura transformations of KP hierarchy, on one hand, and the discrete Toda-like integrable models on refined lattices, on the other hand, will also allow to make further progress in understanding the integrability of the quantum theories.

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